Uncertainty and Error (6/30/22, Dr. Guess)

Introduction

Every measurement comes with associated uncertainty (usually called error). Uncertainty represents how confident one is in the accuracy of the measurement. The two main categorizations are

Systematic: The readings are off in a consistent pattern. An example of a systematic error would be to make a series of measurements with a scale that wasn't properly zeroed (all measurements would be too high or too low by that set amount). Alternatively, a warped meter stick is slightly too short because it's curved. Using it to measure the length of a room would produce a measurement that is too small. In both of these examples, the error is related to the experimental equipment or experimental procedure.

Statistical: The readings vary with every measurement and are equally likely to be too small or too large – these result from the measurement process. An example of a statistical error is to measure a length using the same meter stick. If four successive measurements are taken, they might be

Measurement (cm)	Uncertainty (cm)
75.0	± 1.0
74.5	± 0.5
75.5	± 0.2
75.2	± 0.5

The first listed measurement should be read as: the "best" estimated value is 75.0 cm, but the actual value could be anywhere between 74.0 cm and 76.0 cm.
Multiple successive readings can be slightly different from each other, and the values and their uncertainties can be combined to find a single overall measurement and uncertainty. This procedure is discussed below.

In most introductory physics labs, the uncertainty of a single measurement is quite literally how well you think you measured it!

In a complicated experiment, you should think about all possible sources of error and how it can be reduced. Eliminating uncertainty is not possible. **Please do not cite "human error" in your laboratory results**: everyone makes mistakes, but you should be as specific as possible when talking about them in a laboratory setting.

Percent Error

The standard way to write errors is using the "best" measured value plus or minus the estimated uncertainty. For a measured quantity A, in mathematical notation this reads $A \pm \delta A$ where δA is the uncertainty. However, it is sometimes helpful to think about the uncertainty in terms of the percentage of the measured value A. To calculate percent error, divide δA by A and multiply by 100:

$$\frac{\delta \mathrm{A}}{\mathrm{A}} \times 100 = \%\ \text{error}$$

To go from percent error to standard error, divide by 100 and multiply by A:

$$\frac{\%\ \text{error}}{100} \times A = \delta \mathrm{A}$$

Combining Uncertainties

Most of the uncertainties you deal with as an undergraduate will be statistical. For that reason, the discussion and charts below focus on the proper treatment of statistical errors.

Percentage Errors in multiplication and division (assuming the errors are small)

If the errors are small and two quantities are multiplied or divided, convert the standard errors of the two values into percent errors and add the percent errors. Then you can convert the total percent error back to an overall standard error.

Calculations with Standard Errors

For the calculations below, A and B are measured values and Z is the combined answer. Each measured value has its own uncertainty δA and δB, and we are looking for δZ.

Type of calculation	Equation	Relation between standard errors
Addition/Subtraction	Z = A+B or Z = A-B	$(\delta Z)^2 = (\delta A)^2 + (\delta B)^2$
Multiplication/Division	Z = A*B or A/B	$\left(\frac{\delta Z}{Z}\right)^2 = \left(\frac{\delta A}{A}\right)^2 + \left(\frac{\delta B}{B}\right)^2$
A to a power	$Z = A^n$	$\frac{\delta Z}{Z} = n\frac{\delta A}{A}$
Natural Log of A	Z = ln(A)	$\delta Z = \frac{\delta A}{A}$
Exponential of A	$Z = e^A$	$\frac{\delta Z}{Z} = \delta A$

For cases where you have an equation of several variables that doesn't quite fit one of the above, the most general form for error propagation of independent statistical errors for a function Z(A,B. . .C) is

$$\delta Z = \sqrt{\left(\frac{\partial Z}{\partial A}\delta A\right)^2 + \left(\frac{\partial Z}{\partial B}\delta B\right)^2 + \ldots\left(\frac{\partial Z}{\partial C}\delta C\right)^2}$$

The expression $\frac{\partial Z}{\partial A}$ means to take a partial derivative of function Z with respect to variable A. If we need this general formula, we should go through the calculation as a class.

Example Calculation for multiple statistical measurements (no weighting)

Consider the four independent measurements of length. The average or "mean" value is calculated by adding the individual measurements up and dividing by the number of measurements taken. Our sample calculation looks like:

$$Z = \frac{1}{4}(A + B + C + D) = \frac{1}{4}(75.0cm + 74.5cm + 75.5cm + 75.2cm) = \frac{300.2cm}{4} = 75.05\ cm$$

(If we wanted to be picky about significant figures, we should round it up to 75.1 cm. But let's assume we are going to use this in a follow-up calculation, in which case we want to avoid rounding until the end of the problem.)

The uncertainty calculation is therefore

$$(\delta Z)^2 = (\delta A)^2 + (\delta B)^2 + (\delta C)^2 + (\delta D)^2 = (1cm)^2 + (0.5cm)^2 + (0.2cm)^2 + (0.5cm)^2$$

So our value for δZ is $\delta Z = \sqrt{(1cm)^2 + (0.5cm)^2 + (0.2cm)^2 + (0.5cm)^2} = \sqrt{2.5\ cm^2} = 1.24cm$

And our final value for the measurement is 75.05 ± 1.24 cm.

References

An Introduction to Error Analysis, John R. Taylor. University Science Books, 2nd edition, 1997

Practical Physics, G.L. Squires. Cambridge University Press, 4th edition, 2001